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THE UNSTABLE PLANE MOTION OF THE POISEUILLE TYPE FOR RIVLIN- ERIKSEN FLUID*

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Exact unsteady solutions are sought, which describe a plane flow of the Poiseuille type for the Rivlin- Eriksen fluids for arbitrary pressure drop. All considered here flows become, after infinitely long times, a plane steady Poiseuille flow. In particular cases the results obtained here coincide with those obtained earlier in /1/. Certain statements formulated in /1/ without proof are strictly proved here.

The unsteady plane flow of the Poiseuille type for Rivlin-Eriksen fluid between planes $y=\pm 1$ is defined by the equations with initial and boundary conditions

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{1}{R}\frac{\partial u^2}{\partial y^2} + S\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial t}\right), \tag{1}$$

$$0 = -\frac{\partial p}{\partial y} + 2(2S + S_1)\frac{\partial u}{\partial y}\frac{\partial u^2}{\partial y^2}$$

$$u(y,t) = 0;\ -1 \leqslant y \leqslant 1,\ t \leqslant 0; \tag{2}$$
$$u(y,t) = 0,\ t > 0,\ y = \pm 1$$

where $(u(y,t),0)$ is the dimensionless velocity, $p=p(x,y,t)$ is the pressure, $R>0$ is the Reynolds number, $S>0$ is the parameter of viscoelasticity, and $uS_1>0$ is the lateral velocity parameter. From the second of Eqs.(1) follows that $\partial p/\partial y$ is a function of only y and t and, consequently, $p = f_1(y,t)x + f(y,t)$. Substituting this expression into the first of Eqs.(1), we stipulate $f_1(y,t) = -f(t)$, hence $p=-f(t)x+g(y,t)$ when x is the axis parallel to the walls.

On the assumption that function $f(t)$ is given, we define velocity u by the first of Eqs.(1) and conditions (2). After that pressure $\partial p/\partial y$ is determined by the second of Eqs.(1).

The expression for p implies that the flow must be considered in a bounded region. Since the drop of controlling pressure depends on the motion of fluid, it is reasonable to determine it as in classic Newtonian case.

The velocity u satisfies conditions (2) and the first of Eqs.(1) in which $\partial p/\partial x = f(t)$. Restricting the analysis to the consideration of velocities with exponential damping as $t\to\infty$, by the method of Laplace transformation, we obtain

$$u(y,t) = \int_0^t f(\tau) H(y, t-\tau)\, d\tau,\quad t \in [0,\infty],\quad y \in [-1,1] \tag{3}$$

$$H(y,t) = \sum_{n=0}^{\infty} \frac{4(-1)^n (1+\lambda_n RS)}{\pi(2n+1)} \cos\left[\frac{\pi}{2}(2n+1)y\right] \exp(\lambda_n t)$$

$$\lambda_n = -\frac{\pi^2(2n+1)^2}{4R + \pi^2(2n+1)^2 RS}$$

Since $0 < 1 + RS\lambda_n < 1,\ \lambda_n < 0,$ the series for $H(y,t)$ is uniformly convergent. Hence it is possible in formula (3) to exchange the signs of the integral and sum. Note that the uniform convergence not only of the series determining H but, also, of other series that occur below, can be proved on the basis of properties of λ_n.

From the theory of Laplace transformations we have

$$\lim_{t\to\infty} u(y,t) = \lim_{\lambda\to 0} \lambda \bar{u}(y,\lambda) = \lim_{\lambda\to 0}[\lambda(\overline{f(\lambda)})]\frac{R}{2}(1-y^2) = \frac{R}{2}(1-y^2) f(\infty)$$

From this follows that all obtained here solutions approach the Poiseuille classic solution.

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It is known that in the case of plane steady Poiseuille flow for Newtonian fluids it is possible to specify either the pressure drop $\partial p/\partial x = -f(\infty)$ or the friction $(\partial u/\partial y)_{y=\pm 1} = \mp Rf(\infty)$ at the walls.

Similarly in the Newtonian case $u(y, t)$ is uniquely determined by either $\partial p/\partial x$ or $(\partial u/\partial y)_{y=\pm 1}$ From formula (3) we obtain

$$\left(\frac{\partial u}{\partial y}\right)_{y=\pm 1} = \pm \sum_{n=0}^{\infty} 2(1 + \lambda_n RS) \int_0^t f(\tau) \exp[\lambda_n (t-\tau)]\, d\tau =$$

$$\pm 2 \int_0^t f(\tau) M(t-\tau)\, d\tau$$

$$M(t-\tau) > 0$$

For a specified friction at the walls the obtained results is an equation of the Volterra type, where f is a known function. As $M(t-\tau) > 0$ for any $t > 0$, $(\partial u/\partial y)_{y=\pm 1} = 0$ for $t = 0$, while M and $(\partial u/\partial y)_{y=\pm 1}$ are differentiable functions with respect to t in any interval $(0, a)$, $a > 0$, we conclude, in conformity with the Volterra theorem, that each of these equations has a unique solution. It is clear that for a given function $f(t)$ we have a unique functions defining friction at the wall.

Let us now consider the particular case of $f(t) = P(1 - \exp(-\omega t))$, $P, \omega > 0$, in which solution (3) becomes the solution given in /1/. Representing function u in the form of suitable Fourier series, we obtain the following formula:

$$u(y, t) = \frac{P}{\omega}\left[1 + \frac{\cos hy}{\cos h}\right](\exp(-\omega t) - 1) + \tag{4}$$

$$\sum_{n=0}^{\infty} 16(-1)^{n-1} \frac{PR\omega \cos[\pi(n+1/2)y]}{(2n+1)^3 \pi^3 (\omega + \lambda_n)} (\exp(\lambda_n t) - 1)$$

$$h = \sqrt{\frac{R\omega}{1 - RS\omega}}, \quad y \in [-1, 1], \quad t \in [0, \infty)$$

To obtain expressions for derivatives of u it is necessary to investigate addtionally their behavior at points $y = \pm 1$, where the respective Fourier series do not always converge to the expanded function.

To find the derivatives we used the technique of reverse integration in conformity with /2/. It follows from formula (4) that

$$\frac{\partial u}{\partial y} = \frac{P}{\omega} h \frac{\sin hy}{\cos h} (\exp(-\omega t) - 1) - \tag{5}$$

$$\sum_{n=0}^{\infty} 8(-1)^{n-1} \frac{PR\omega \sin[(n+1/2)y]}{(2n+1)^2 \pi^2 (\omega + \lambda_n)} (\exp(\lambda_n t) - 1)$$

$$y \in [-1, 1], \quad t \in [0, \infty)$$

$$p = -P(1 - \exp(-\omega t))x + (2S + S_1)\left(\frac{\partial u}{\partial y}\right)^2 + \text{const}$$

The last expression shows that pressure increases an account of Newtonian terms. Applying formula (3) to the particular cases considered in /3/, we again obtain all the results of /3/.

In the particular cases considered above the numerical computations were carried out for $P = 1$, $S = 1, 2, 5, 10, 100$, $R = 1, 4, 10, 100$, $\omega = 2, 10, 100$ and for different values of t and y.

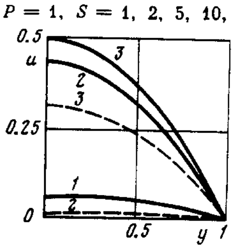

Fig.1

The results of computations are shown in Fig.1 for $P = 1$, $R = 1$, $\omega = 10$ by solid lines for $S = 1$ and by dash lines when $S = 100$. Curves 1–3 correspond to t equal 0.3, 5, and 100. These curves and, also, formulas (3) show that when $t = 0$ the profile of velocity is a straight line $u = 0$, $y \in (-1, 1)$, for $t > 0$, u is a positive increasing function of t which as $t \to \infty$ approaches the Poiseuille parabolic profile. The profile of velocity is symmetric about the x axis. Furthermore u is an increasing function of P and independent of S_1.

Pressure increases with S_1. It will be seen that increasing R or ω and decrease of S results in increasing velocity.

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